ON A CERTAIN MOTION CORRECTION PROBLEM

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The problem of multiple correction of the perturbed trajectory of a controlled object with a restricted controlling force and tracking of some of the coordinates is considered. The problem investigated belongs to the class of problems analyzed in [1−5].

1. Formulation of the problem. Let the motion of a controlled object over a given time interval $0 \leqslant t \leqslant \vartheta$ be described in linear approximation by the vector differential equation

$$dx / dt = Ax + Bu \tag{1.1}$$

where x is the n-dimensional vector of the phase coordinates of the object measured from the prescribed motion; A and B are matrices of the appropriate dimensions; u is the r-dimensional vector of the controlling force whose intensity $\varkappa[u]$ is restricted by the inequality

$$\varkappa\,[u\,(\tau)] \leqslant \mu \qquad (\mu = \mathrm{const} > 0) \tag{1.2}$$

Let us assume that the quantity $\varkappa[u]$ takes the form of some norm of the vector function $u\,(\tau)$, $(t \leqslant \tau \leqslant \vartheta)$ (e.g. see [5], pp. 34, 233).

Our problem is to construct a control u which ensures the minimal error $\varepsilon\,(\vartheta) = \| \{x(\vartheta)\}_m \|$. Here $\{x\}_m$ denotes the set of m isolated phase coordinates x_{i_s} $(s = 1, \ldots, m)$ used to estimate the deviation of the true position of the object at the instant $t = \vartheta$ from the prescribed position. This set of coordinates can be treated as an m-dimensional vector $q = \{q_s\}$ in some space $\{q\}$. The symbol $\|q\|$ denotes the norm of the vector q.

Let us consider the problem of finding the control which minimizes the error ε complicated by the lack of complete information on the initial state $x(0)$ of the object. Let us assume that the deviation $x(0)$ of the object from the nominal trajectory at the initial instant is not known exactly, and that we are merely given the domain of scatter $G(0)$ of the possible phase coordinates of the object for $t = 0$.

Let us assume, moreover, that refinement of information about the instantaneous phase state of the object is assisted by additional tracking, so that by an instant t from the interval $(0, \vartheta)$ we know the values of a pair of vector functions $\{z(\tau), u(\tau)\}$ $(0 \leqslant \tau \leqslant t)$, where the values of the k-dimensional vector $z(\tau)$ $(k < n)$ are related to the phase vector $x(\tau)$ by the expressions

$$z(\tau) = Q(\tau)x(\tau) + \Delta(\tau) \tag{1.3}$$

Here $Q(\tau)$ is some matrix of order $k \times n$; $\Delta(\tau)$ is a measurement error whose intensity $\chi[\Delta]$ is restricted by the condition

$$\chi\,[\Delta(\tau)] \leqslant \nu, \ (0 \leqslant \tau \leqslant t,\ \nu > 0 - \mathrm{const}) \tag{1.4}$$

Here too we assume that the intensity $\chi\,[\Delta(\tau)]$ of the error can be interpreted as some norm of the vector function $\Delta(\tau)$ $(0 \leqslant \tau \leqslant t)$.

Finally, let us assume that we have already chosen some instants $t = t_j\,(t_j < \vartheta)$ $(j = 1, \ldots, l)$ such that the control $u\,(t)$ in each interval $(t_j,\ t_{j+1})$ can be determined on the basis of the tracking data for $0 \leqslant \tau \leqslant t_j$.

Let us refine our picture of the controlled process. Let us assume that data obtained by monitoring the signal $\{z(\tau),\ u(\tau)\}$ $(0 \leqslant \tau \leqslant t_j)$ have indicated that at the instant $t = t_j$ the values of the phase vector $x(t_j)$ belong to the domain $G(t_j)$. The control $u^{(j)}(t)$ in the time interval $t_j < t \leqslant t_{j+1}$ is then constructed as follows: we find a control $u^*(t)$ $(t_j \leqslant t \leqslant \vartheta)$ which ensures that

$$J(G(t_j)) = \min_u \max_x \varepsilon_u(\vartheta) = \max_x \varepsilon_{u^*}(\vartheta) \qquad (1.5)$$
$$x(t_j) \in G(t_j), \qquad \varkappa[u] \leqslant \mu$$

and set $u^{(j)}(t) = u^*(t)$ for $t_j < t \leqslant t_{j+1}$. This control guarantees a true miss $\varepsilon(\vartheta) \leqslant J(G(t_j))$; it is obvious, moreover, that $J(G(t_{j+1})) \leqslant J(G(t_j))$. We assume that the domain $G(0)$ is given. The domain $G(t_j)$ is found by monitoring the signal $\{z(\tau), u(\tau)\}$ $(0 \leqslant \tau \leqslant t_j)$, and the specific form of this domain clearly depends on the specific tracking operations $\varphi^{(j)} = \varphi[t_j,\ \{z(\tau),\ u(\tau)\}]$ performed on the signals $\{z(\tau), u(\tau)\}$, $(0 \leqslant \tau \leqslant t_i;\quad i = 1,\dots, j)$ during motion (1.1).

The magnitude of the miss $J(G(t_j))$ given by (1.5) therefore depends on the tracking operations, i.e. $J(G(t_j)) = f(\varphi[t_1,\ \{z(\tau),\ u(\tau)\}],\dots,\ \varphi[t_j,\ \{z(\tau),\ u(\tau)\}])$ (1.6)

This implies, in turn, that the operations $\varphi^{(i)}$ should be chosen in such a way as to minimize the quantity $J(G(t_j))$ over all the possible operations $\varphi^{(i)}$. The above control procedure then yields the optimal result $\varepsilon(\vartheta)$. However, the procedure of minimizing $f(\varphi^{(1)},\dots,\varphi^{(j)})$ over the operations $\varphi^{(i)}$ is a difficult one if the class of permissible operations $\varphi^{(i)}$ is broad.

In order to avoid this difficulty we shall restrict the set of permissible tracking operations $\varphi^{(i)}$ as follows. Let $t = t_j$ be some instant and let $X[t,t_j]$ be the fundamental matrix of homogeneous system (1.1) for $u \equiv 0$ $(X[t_j,t_j] = E)$, and let the domain $G(0)$ be such that the linear transformation $q = \{X[\vartheta,\ 0]x\}_m$ maps the domain $G(0)$ from the n-dimensional space $\{x\}$ into some rectangular parallelepiped $G_q(0)$ from the m-dimensional space $\{q\}$; let the faces of this parallelepiped be parallel to the corresponding coordinate planes.

If this condition is not fulfilled in the initial problem, then we can imbed $G(0)$ in a larger domain $G'(0)$ which satisfies this assumption. In accordance with this condition we shall consider only those tracking operations which at each instant $t = t_j$ define a domain $G(t_j)$ in $\{x\}$ such that its $X[\vartheta,\ t_j]$-image in $\{q\}$ $(q(t_j) = \{X[\vartheta, t_j]G(t_j)\}_m)$ is also a rectangular parallelepiped (Fig. 1).

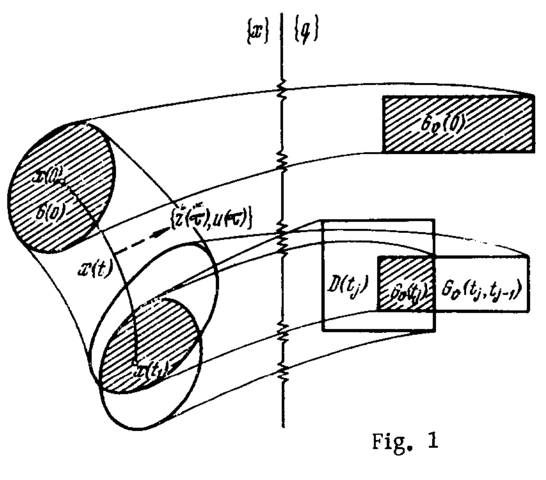

Fig. 1

The range of such operations is not difficult to determine. These operations $\varphi[t_j,\ \{z(\tau),\ u(\tau)\}]$ will be such that their components $\varphi_s[t_j,\ \{z(\tau), u(\tau)\}]$ $(s = 1,\dots, m)$ solve the tracking problem (e.g. see [5], p. 293] of the linear function

$$q_s(t_j) = h^{[i_s]}[\vartheta,\ t_j]\, x(t_j), \qquad (s = 1, \dots, m) \qquad (1.7)$$

on the basis of the signal $\{z(\tau), u(\tau)\}$ $(0 \leqslant \tau \leqslant t_j)$. (Here $h^{[i_s]}[\vartheta,\ t_j]$ is the vector row of the matrix $X[\vartheta, t_j]$ corresonding to the i_sth coordinate of the phase vector x.) We know from optimal tracking theory [5] that among the operations φ_s which compute the values of the coordinates $q_s(t_j)$ (1.7) of the vector $q(t_j)$ there exists an optimal resolving operation φ_s° which computes the quantity $q_s(t_j)$ (1.7) with the smallest possible error in the least favorable case of the signal $\{z(\tau), u(\tau)\}$. In other words, for every other operation $\varphi_s^{(j)}$ which can be used at the instant $t = t_j$ we have the relation

$$\sup_z |\varphi_s^{\circ(j)} - q_s(t_j)| = \min_\varphi \sup_z |\varphi_s^{(j)} - q_s(t_j)| \tag{1.8}$$

The operation $\varphi^{\circ(j)}$ therefore defines some domain $D(t_j)$ in the space $\{q\}$ which is described by the inequalities $|q_s^*(t_j) - q_s(t_j)| \leqslant \delta_s(t_j) \qquad (s = 1, \ldots, m)$
where $\delta_s(t_j)$ is the upper bound of the absolute value of the error of computing the true value of the coordinate $q_s(t_j)$ with respect to all the possible disturbances $\Delta(\tau)$ (1.3), (1.4); $q_s^*(t_j)$ is the value of the coordinate of the vector $q(t_j)$ computed at the instant $t = t_j$ by the operation $\varphi_s^{\circ(j)}$.

In computing $J(G(t_j))$ (1.6) we must also take account of the results of the previous measurements in the interval $0 \leqslant \tau \leqslant t_i$ $(i = 1, \ldots, j)$ and the realized control $u(t)$ $(0 \leqslant t \leqslant t_j)$. We make these allowances by assuming that as a result of the last observation made at the instant $t = t_{j-1}$ we have determined the domain $G_\vartheta(t_{j-1})$ in the form of a rectangular parallepiped (Fig. 1) in the vector space $\{q\}$.

Since system (1.1) moved under the control $u^{(j-1)}(t) = u^*(t)$, in the interval $t_{j-1} < t \leqslant t_j$, it follows that the domain $G_\vartheta(t_{j-1})$ is deformed into the domain $G_\vartheta(t_j, t_{j-1})$ whose points $q(t_j, t_{j-1})$ are given by Eqs.

$$q(t_j, t_{j-1}) = q(t_{j-1}) + \left\{X[\vartheta, t_j] \int_{t_{j-1}}^{t_j} X[t_j, \tau] B u^{(j-1)}(\tau)\, d\tau\right\}_m =$$
$$= q(t_{j-1}) + g(t_j, t_{j-1}), \quad q(t_{j-1}) \in G_\vartheta(t_{j-1})$$

Thus, the domain $G_\vartheta(t_j, t_{j-1})$ is again a rectangular parallelepiped, since the points of the domain $G_\vartheta(t_j, t_{j-1})$ are obtainable by a shift by the same vector $g(t_j, t_{j-1})$. But if we now allow for the result of tracking in the interval $0 \leqslant \tau \leqslant t_j$, we see that the domain $G_\vartheta(t_j)$ of possible values $q(t_j)$ computed on the basis of the signal $z(\tau)$ $(0 \leqslant \tau \leqslant t_j)$ is the set $G_\vartheta(t_j) = G_\vartheta(t_j, t_{j-1}) \cap D(t_j)$. It is clear that the domain $G_\vartheta(t_j)$ is also a rectangular parallelepiped with faces parallel to the coordinate planes in the vector space $\{q\}$.

Although this approach may not yield a strictly optimal control result, and even though our procedure provides only a relatively minimal miss $J(G(t_j))$ (1.6), all of its computations are at least practically feasible. Thus, the procedure of constructing a control at each instant $t = t_j$ breaks down into two subsidiary problems: the problem of choosing the tracking operations $\varphi_s^\circ[t_j, \{z(\tau), u(\tau)\}]$ which compute the quantities $q_s(t_j)$ (1.7) (which essentially reduces to the problem of determining the domain $G_\vartheta(t_j)$), and the problem of determining the minimal miss $J(G(t_j))$ and the control $u^{(j)}(t)$ which ensures this miss.

2. Let us assume that the set $G_\vartheta(t_j)$ of vectors $q(t_j = \{X[\vartheta, t_j] x(t_j)\}_m$, $(x(t_j) \in G(t_j))$ corresponding to some domain $G(t_j)$ of possible states $x(t)$ of the object at $t = t_j$ has been determined for the instant $t = t_j$. Let us consider the following ancillary problem.

Let the motion of the object in the time interval $t_j \leqslant t \leqslant \vartheta$ be described by Eqs. (1.1), (1.2). We are to choose a control $u^*(t)$ from among permissible controls (1.2) in such a way as to ensure that

$$J = \min_u \max_x \varepsilon(\vartheta) \quad \text{for} \quad \varkappa[u] \leqslant \mu, \; x \in G(t_j) \tag{2.1}$$

This is therefore the problem which must be solved at each step $t = t_j$ of the correction decision process. To determine the control $u^*(t)$ which solves problem (2.1) we transform expression (1.6) for the miss $J(G(t_j))$. By the Cauchy formula the values of the selected coordinates x_{i_s} at $t = \vartheta$ are given by the relation

$$\{x(\vartheta)\}_m = \{X[\vartheta, t_j]\, x(t_j)\}_m + \left\{\int_{t_j}^{\vartheta} X[\vartheta, \tau]\, Bu(\tau)\, d\tau\right\}_m \tag{2.2}$$

Since $\varepsilon(\vartheta) = \left\| \{x(\vartheta)\}_m \right\|$, expression (2.1) with allowance for (2.2) becomes

$$J(G(t_j)) = \min_u \max_q \left\| q + \left\{\int_{t_j}^{\vartheta} X[\vartheta, \tau]\, Bu(\tau)\, d\tau\right\}_m \right\| \tag{2.3}$$

$$\varkappa[u] \leqslant \mu, \quad q \in G_\vartheta(t_j)$$

We note that the quantity $w(\vartheta)$ given by

$$w(\vartheta) = \int_{t_j}^{\vartheta} X[\vartheta, \tau]\, Bu(\tau)\, d\tau$$

is the solution of Eq.

$$dw/dt = Aw + Bu, \quad w(t_j) = 0 \tag{2.4}$$

Let us write

$$\max_q \| q + \{w\}_m \| = \gamma[\{w\}_m], \qquad q \in G_\vartheta(t_j) \tag{2.5}$$

Then

$$J(G(t_j)) = \min_u \gamma[\{w(\vartheta)\}_m] \quad \text{for} \quad \varkappa[u] \leqslant \mu, \; t_j \leqslant \tau \leqslant \vartheta \tag{2.6}$$

We have therefore reduced problem (2.1), (1.2) of determining the control $u^*(t)$ $(t_j \leqslant t \leqslant \vartheta)$ which guarantees the smallest miss $J(G(t_j))$ for system (1.1) with the initial conditions $x(t_j)$ from $G(t_j)$ to the problem of bringing system (2.4) from the point $w(t_j) = 0$ to some point $w^\circ(\vartheta)$ corresponding to the minimum of the function $\gamma[\{w(\vartheta)\}_m]$.

Solution of problem (2.3)–(2.5) in this case is facilitated by the convenient properties of the set $Q_\zeta(\{w\}_m)$ of points w satisfying the condition

$$Q_\zeta(\{w\}_m) \equiv \{\{w(\vartheta)\}_m : \; \gamma[\{w(\vartheta)\}_m] \leqslant \zeta, \; \zeta = \text{const} > 0\}$$

We can show that the sets $Q_\zeta(\{w\}_m)$ are convex and closed. Let $\{w^{(1)}\}_m$ and $\{w^{(2)}\}_m$ be some points from the set $Q_\zeta(\{w\}_m)$. This means that

$$\gamma[\{w^{(1)}\}_m] = \max_q \| q + \{w^{(1)}\}_m \| \leqslant \zeta$$

$$\gamma[\{w^{(2)}\}_m] = \max_q \| q + \{w^{(2)}\}_m \| \leqslant \zeta, \quad q \in G_\vartheta(t_j)$$

Let us choose a point

$$\{w^{(\lambda)}\}_m = \lambda \{w^{(1)}\}_m + (1 - \lambda)\{w^{(2)}\}_m \quad (0 \leqslant \lambda \leqslant 1)$$

from the segment connecting the points $\{w^{(1)}\}_m$ and $\{w^{(2)}\}_m$ from $Q_\zeta(\{w\}_m)$. Then

$$\gamma[\{w^{(\lambda)}\}_m] = \max_q \| q + \{w^{(\lambda)}\}_m \| = \| q^\lambda + \{w^{(\lambda)}\}_m \| = \| \lambda (q^\lambda + \{w^{(1)}\}_m) +$$
$$+ (1 - \lambda)(q^\lambda + \{w^{(2)}\}_m) \| \leqslant \lambda \| q^\lambda + \{w^{(1)}\}_m \| + (1 - \lambda) \| q^\lambda + \{w^{(2)}\}_m \| \leqslant \lambda\zeta + (1 - \lambda)\zeta = \zeta,$$
$$q \in G_\vartheta(t_j)$$

This proves that the entire segment belongs to the set $Q_\zeta(\{w\}_m)$, i.e. that the set

$Q_\zeta(\{w\}_m)$ is convex. It is also easy to verify the fact that $Q_\zeta(\{w\}_m)$ is a closed set.

Since the sets $Q_\zeta(\{w\}_m)$ are convex and closed, it follows (see [5], p. 314) that the quantity $\min_u \gamma[\{w\}_m] = \zeta^\circ$ is the smallest of the numbers ζ satisfying the condition

$$\max\{\gamma_\zeta^*[k] - \mu\rho[B'S[\tau, \vartheta]k]\} \leqslant 0, \qquad \|k\| = 1 \tag{2.7}$$

$$\gamma_\zeta^*[k] = \min w' \cdot k \qquad \text{for } w \in Q_\zeta(\{w\}_m)$$

Here $S[\tau, \vartheta]$ is the fundamental matrix of the system $\dot{s} = -A's$ associated with system (1.1) for $u \equiv 0$. Further, $\rho[h(\tau)]$ is the norm in the space $B\{h(\tau),\ t_j \leqslant \tau \leqslant \vartheta\}$ of the vector functions $h(\tau)$ associated with the space $B^*\{u(\tau),\ t_j \leqslant \tau \leqslant \vartheta\}$ with the norm $\rho^*[u(\tau)] = \varkappa\ [u(\tau)]$ (1.2).

If condition (2.7) for $\zeta = \zeta^\circ = \min$ is fulfilled with the equality sign, then the optimal control $u^*(t)$ which solves problem (2.3)–(2.5) can be found from the maximum condition (see [5], p. 314)

$$\int_{t_j}^{\vartheta} k^{\circ\prime} S[\tau, \vartheta] B u^*(\tau)\, d\tau = \max_u \int_{t_j}^{\vartheta} k^{\circ\prime} S[\tau, \vartheta] B u(\tau)\, d\tau \tag{2.8}$$

$$\rho^*[u] \leqslant \mu$$

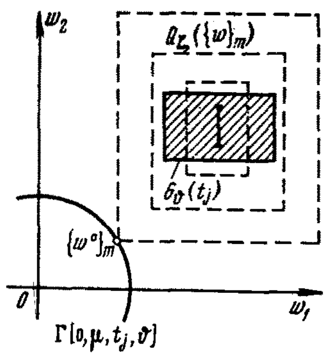

Fig. 2

This relation has the following geometric significance. Let $\Gamma[0, \mu, t_j, \vartheta]$ denote the attainability domain ([5], p. 116) of the process $w(t)$ (2.4) in the space $\{q\}$, i.e. the domain consisting of all the points $q = \{w(t)\}_m$ to which it is possible to bring the motion $w(t)$ (2.4) by the instant $t = \vartheta$ by means of the control $u(t)$ ($t_j \leqslant t \leqslant \vartheta$) subject to condition (1.2). Fulfillment of inequality (2.7) means that the domain $\Gamma[0, \mu, t_j, \vartheta]$ has points $\{w\}_m$ in common with the set $Q_\zeta(\{w\}_m)$. If equality applies in (2.7), then the domains $\Gamma[0, \mu, t_j, \vartheta]$ and $Q_\zeta(\{w\}_m)$ are merely adjacent. The point $\{w^\circ\}_m$ from $\Gamma[0, \mu, t_j, \vartheta]$ corresponding to the minimum of the function $\gamma[\{w\}_m]$ then lies on the boundary of the attainability domain (Fig. 2), and the control $u^*(t)$ aims the motion $w(t)$ at the point $\{w^\circ\}_m$.

If condition (2.7) is an inequality, then the point where the function $\gamma[\{w\}_m]$ reaches its absolute minimum lies inside the attainability domain $\Gamma[0, \mu, t_j, \vartheta]$ (Fig. 3). In this case we can alter μ in such a way that equality once again applies in (2.7) and then proceed to find u from maximum condition (2.8).

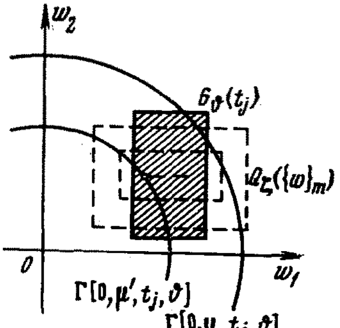

Fig. 3

3. Solution of the tracking problem.

Let us describe briefly the solution of the optimal tracking problem involving determination of the quantities $\delta_s(t_j)$ and $D(t_j)$ ([5], p. 247). We assume that the vector functions $z(\tau)$, $\Delta(\tau)$ and $y(\tau) = Q(\tau)x(\tau)$ are elements $h(\tau)$ of some function space $B\{h(\tau),\ 0 \leqslant \tau \leqslant t_j\}$ in which the norm $\rho[h(\tau)]$ is defined by the equation

$\rho\,[h(\tau)] = \chi\,[h(\tau)]$ (1.4). We are required to compute the vector $q = \{X\,[\vartheta,\ t_j]\,x(t_j)\}_m$

Let us consider the computation of some single coordinate q_s of this vector. We assume that s is fixed. We can determine all the possible linear bounded operations $\varphi_s\,[t_j,\ \{z(\tau),\ u(\tau)\}]$ in the space $B\,\{h(\tau),\ 0 \leqslant \tau \leqslant t_j\}$ from which we must choose the operation φ_s° which computes the quantity $q_s(t_j)$ with the smallest possible error $\omega_s(t_j) = \varphi_s^\circ\,[t_j,\ \{z(\tau),\ u(\tau)\}] - q_s(t_j)$ satisfying condition (1.8).

Let us proceed as in [5], p. 294. We begin by setting $u \equiv 0$ and $\Delta \equiv 0$ and considering the signals

$$y(\tau) = Q(\tau)\,X\,[\tau,\ t_j]\,x(t_j) \qquad (0 \leqslant \tau \leqslant t_j) \tag{3.1}$$

Let us find the operation φ_s° which reconstructs the values of the coordinates $q_s = [\{X\,[\vartheta\ t_j]\,x(t_j)\}_m]_s$. According to the minimax rule ([5], p. 285), this can be done by finding from among the signals $y(\tau)$ (3.1) the signals $\{y(\tau)\,|\,q_s(t_j) = 1\}$ which carry a $q_s(t_j)$ equal to unity. These signals can be found from the condition

$$\{y(\tau)\,|\,q_s(t_j) = 1\} = [Q(\tau)\,X\,[\tau,\ t_j]\,x(t_j)]_{q_s(t_j)=1}$$

The required operation $\varphi_s\,[t_j,\ y(\tau)]$ must satisfy the condition

$$\varphi_s\,[t_j,\ \{y(\tau)\,|\,q_s(t_j) = 1\}] = 1 \tag{3.2}$$

on any signal $\{y(\tau)\,|\,q_s(t_j) = 1\}$ carrying the quantity $q_s(t_j) = 1$. Knowing the signals $\{y(\tau)|\,q_s(t_j) = 1\}$, we must find the minimum signal $y^\circ(\tau)$ from the condition

$$\chi_s^\circ = \chi\,[y^\circ(\tau)] = \min_y \chi\,[\{y(\tau)\,|\,q_s(t_j) = 1\}] \tag{3.3}$$

By the minimax rule, the optimal resolving operation φ_s° has the norm

$$\chi^*\,[\varphi_s^{\circ(j)}] = \chi^*\,[\varphi_s^\circ\,[t_j,\ y(\tau)]] = 1/\chi_s^\circ \tag{3.4}$$

and is distinguishable from the other linear operations φ_s in that this operation on the minimum signal $y^\circ(\tau)$ yields the largest possible result as compared with all the other operations φ_s with the same norm (3.4), i.e.

$$\varphi_s^\circ\,[t_j,\ y^\circ(\tau)] = \max_\varphi \{\varphi_s\,[t_j,\ y^\circ(\tau)] \quad \text{for} \quad \chi^*\,[\varphi_s] = 1/\chi_s^\circ\} \tag{3.5}$$

Conditions (3.4)–(3.5) define the operation φ_s°. For $u \equiv 0$ this operation also solves our optimal tackling problem with respect to the signal $z(\tau) = y(\tau) + \Delta(\tau)$. Only in the case of a real signal $z(\tau)$ does the operation φ_s° obtained from condition (3.2)–(3.5) yield an unavoidable error $\omega_s(t_j)$, where ([5], p. 281)

$$\sup_\Delta |\,\omega_s(t_j)\,| = \sup_\Delta |\,\varphi_s^\circ\,[t_j,\ \Delta(\tau)]\,| = \nu\chi^*\,[\varphi_s^\circ] = \nu/\chi_s^\circ = \delta_s(t_j)$$

In the case where the signal $\{z(\tau),\ u(\tau)\}$ is tracked for $u \not\equiv 0$, the required operation $\varphi_s^\circ\,[t_j,\ \{z(\tau),\ u(\tau)\}]$ which reconstructs the quantity $q_s(t_j)$ from the signal $\{z(\tau),\ u(\tau)\}$ is easy to determine if we know the operation $\varphi_s^\circ\,[t_j,\ y(\tau)]$ for $u \equiv 0$. Since any motion $x(\tau)$ of system (1.1) can be found from Cauchy formula (2.2) (with τ appearing instead of ϑ), we have

$$y_*(\tau) = Q(\tau)\,X\,[\tau,\ t_j]\,x(t_j) + Q(\tau)\int_{t_j}^{\tau} X\,[\tau,\ \zeta]\,Bu(\zeta)\,d\zeta = y(\tau) + Q(\tau)\int_{t_j}^{\tau} X\,[\tau,\ \zeta]\,Bu(\zeta)\,d\zeta$$

for $u \not\equiv 0$.

Recalling the linearity of the operation φ_s°, we obtain

$$\varphi_s^\circ\,[t_j,\ \{y_*(\tau),\ u(\tau)\}] = \varphi_s^\circ\,[t_j,\ y(\tau)] - \varphi_s^\circ\,[t_j,\ v(\zeta)] \tag{3.6}$$

$$v(\zeta) = Q(\zeta)\int_{t_j}^{\zeta} X\,[\zeta,\ \tau]\,Bu(\tau)\,d\tau$$

Operation (3.6) is the optimal resolving operation which computes the coordinate $q_s(t_j)$ from the signal $\{z(\tau),\ u(\tau)\}$ with the smallest possible error $\sup_\Delta |\omega_s(t_j)| = \min_\varphi$.

4. Example. Let us consider a material point m moving in some plane x_1x_3 under the action of a repelling force proportional to the distance from the point m to the origin $(x_1 = 0, x_3 = 0)$. Let us assume that the motion of the point is described in this case by the differential equations

$$x_1^\cdot = x_2, \qquad x_2^\cdot = x_1 + u_1, \qquad x_3^\cdot = x_4, \qquad x_4^\cdot = x_3 + u_2 \tag{4.1}$$

Here $\{x_1, x_3\}$ is the radius vector of the point, $\{x_2, x_4\}$ is its velocity vector, and $\{u_1, u_2\}$ is the controlling force whose purpose is to minimize the deviation

$$J = \max(|x_1(\vartheta)|,\ |x_3(\vartheta)|) \tag{4.2}$$

of the point from the origin at the given instant $t = \vartheta$.

The coordinates and velocity of the point at the initial instant $t = 0$ are assumed to be unknown; all that we know is the domain

$$G_\vartheta(0) \equiv \{\{q_s\}:\ a_1^{(0)} \leqslant q_1 \leqslant b_1^{(0)},\ \ a_2^{(0)} \leqslant q_2 \leqslant b_2^{(0)}\}$$

which the point m can have entered by the instant $t = \vartheta$ in the absence of the control $(u \equiv 0)$. We assume, moreover, that measurement and memorization of the magnitude and direction of the velocity at each instant t are possible during motion. The velocity can only be measured to within some error whose exact value is not known. We assume in advance, however, that the absolute value of the error Δ_i involved in measuring each of the velocity components x_2 and x_4 cannot exceed some constant ν $(\nu > 0)$, so that

$$\eta[\Delta] = \max_\tau \{\max(|\Delta_1(\tau)|,\ |\Delta_2(\tau)|)\} \leqslant \nu \qquad (0 \leqslant \tau \leqslant t) \tag{4.3}$$

We are required to choose a control program which uses the measured velocity of the point and the known law of variation of the controlling force u as the sole basis for ensuring that the point approaches the origin by the instant $t = \vartheta$. We assume that the control program can be altered only at preselected instants $t_0, t_1, \ldots, t_l$ (where $t_0 = 0$ is the initial instant of the point's motion). The magnitude of the correcting control u must not exceed a prescribed value μ, i.e. the following condition must be fulfilled throughout the time of motion:

$$\|u(t)\|^2 = u_1^2(t) + u_2^2(t) \leqslant \mu^2, \qquad 0 \leqslant t \leqslant \vartheta, \qquad \mu = \text{const} \tag{4.4}$$

Let $t = t_j$ be some instant of motion correction at which it is necessary to switch to the control $u^{(j)}(t)$. To determine the control $u^{(j)}(t)$ we must construct the domain $G_\vartheta(t_j)$. Let us begin by finding the operations $\varphi_s^\circ[t_j, \{z(\tau), u(\tau)\}]$ which reconstruct the coordinates $q_s(t_j)$ (1.7) of the vector $q(t_j)$ at the instant $t = t_j$. We note that system (4.1) (for $u \equiv 0$) is quite trackable with respect to velocity, so that such operations exist.

By the condition of the problem the monitored signal $z(\tau)$ (1.3) can be written in the form

$$\begin{pmatrix} z_1(\tau) \\ z_2(\tau) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(\tau) \\ \cdots \\ x_4(\tau) \end{pmatrix} + \begin{pmatrix} \Delta_1(\tau) \\ \Delta_2(\tau) \end{pmatrix}$$

Recalling that the matrix $X[t, \tau]$ given by

$$X[t, \tau] = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \qquad c = \begin{pmatrix} \operatorname{ch}(t-\tau) & \operatorname{sh}(t-\tau) \\ \operatorname{sh}(t-\tau) & \operatorname{ch}(t-\tau) \end{pmatrix}$$

is the fundamental matrix of system (4.1) for $u \equiv 0$, we obtain the following expression for $q_s(t_j)$ (1.7):

$$q_s^\cdot(t_j) = x_{2s-1}(t_j)\operatorname{ch}(\vartheta - t_j) + x_{2s}(t_j)\operatorname{sh}(\vartheta - t_j)$$

In the notation of Sect. 3 we have

$$\{y(\tau) \mid q_1 = 1\} = \begin{pmatrix} \alpha_1 \\ \beta_2 \end{pmatrix} \qquad \{y(\tau) \mid q_2 = 1\} = \begin{pmatrix} \beta_1 \\ \alpha_2 \end{pmatrix}$$

$$\alpha_s = [\operatorname{ch}(\tau - t_j) - x_{2s-1}(t_j)\operatorname{ch}(\vartheta - \tau)] / \operatorname{sh}(\vartheta - \tau)$$

$$\beta_s = x_{2s-1}(t_j)\operatorname{sh}(\tau - t_j) + x_{2s}(t_j)\operatorname{ch}(\tau - t_j) \qquad (s = 1, 2)$$

Now let us take the quantity $\eta[\Delta]$ (4.3) as the norm of $\chi[y(\tau)]$ in the space $B\{y(\tau), 0 \leqslant \tau \leqslant t_j\}$. We can then find the signals $\{y^\circ|(\tau)|q_1 = 1\}$ and $\{y^\circ(\tau) \mid q_2 = 1\}$ from the condition

$$\begin{aligned} \chi_1^\circ &= \min_{x_1} \max_\tau [\max(|\alpha_1|, |\beta_2|)] \\ \chi_2^\circ &= \min_{x_2} \max_\tau [\max(|\alpha_2|, |\beta_1|)] \end{aligned} \qquad (0 \leqslant \tau \leqslant t_j)$$

They are of the form

$$\{y^\circ(\tau) \mid q_1 = 1\} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \qquad \{y^\circ(\tau) \mid q_2 = 1\} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}$$

$$\xi = \frac{\operatorname{sh}(\tau - {}^1/_2 t_j)}{\operatorname{ch}(\vartheta - {}^1/_2 t_j)} \qquad \chi_1^\circ = \chi_2^\circ = \frac{\operatorname{sh} t_j}{\operatorname{ch}\vartheta + \operatorname{ch}(\vartheta - t_j)}$$

The operation $\varphi_s^\circ[t_j, y(\tau)]$ which reconstructs the quantity q_s (in the case of the norm $\chi[\Delta(\tau)] = \eta[\Delta]$ (4.3)) is of the form

$$\varphi_s^\circ[t_j, y(\tau)] = \int_0^{t_j} y_1(\tau)\, dV_{s1}^\circ(\tau) + y_2(\tau)\, dV_{s2}^\circ(\tau)$$

Here $V_{si}(\tau)$ is a function with bounded variation. Recalling that the norm of the operation φ_s° is given by

$$\chi^*[\varphi_s^\circ] = \int_0^{t_j} |dV_{s1}^\circ(\tau)| + |dV_{s2}^\circ(\tau)| = \frac{1}{\chi_s^\circ}$$

and that the values of the operation on the minimal signal is equal to unity, we obtain

$$\varphi_1^\circ[t_j, y(\tau)] = \int_0^{t_j} y_1(\tau)\, dV^\circ_{11}(\tau), \quad \varphi_2^\circ[t_j, y(t)] = \int_0^{t_j} y_2(\tau) d\overset{\circ}{V}_{22}(\tau)$$

$$V_{12}^\circ(\tau) = V_{21}^\circ(\tau) = 0 \qquad (0 \leqslant \tau \leqslant t_j)$$

$$V_{11}^\circ(\tau) = V_{22}^\circ(\tau) = \begin{cases} 0 & (\tau = 0) \\ -\operatorname{ch}(\vartheta - t_j) / \operatorname{sh} t_j & (0 < \tau < t_j) \\ \operatorname{sh}(\vartheta - {}^1/_2 t_j) / \operatorname{ch} {}^1/_2 t_j & (\tau = t_j) \end{cases}$$

The components $v_s(\zeta)$ of the vector $v(\zeta)$ (3.6) are

$$v_s(\zeta) = \int_{t_j}^{\zeta} \operatorname{ch}(\zeta - \tau)\, u_s(\tau)\, d\tau$$

Hence, the operation $\varphi_s^\circ[t_j, \{z(\tau), u(\tau)\}]$, which solves the problem of determining the coordinate $q_s(t_j)$ of the vector $q(t_j)$ from the signal $\{z(\tau), u(\tau)\}$ can be written as

$$\varphi_s^\circ[t_j, \{z(\tau), u(\tau)\}] = \frac{1}{\operatorname{sh} t_j}\Big[x_{2s}(t_j)\operatorname{ch}\vartheta - x_{2s}(0)\operatorname{ch}(\vartheta - t_j) -$$

$$- \operatorname{ch}(\vartheta - t_j)\int_0^{t_j} \operatorname{ch}\tau\, u_s(\tau)\, d\tau\Big] = q_s^*(t_j)$$

This operation yields the error $\omega_s(t_j)$ whose absolute value has the upper bound $\delta(t_j)$ given by

$$\delta(t_j) = \frac{\nu[\operatorname{ch}\vartheta + \operatorname{ch}(\vartheta - t_j)]}{\operatorname{sh} t_j}$$

Thus, the domain $D(t_j)$ is defined by the inequalities

$$D(t_j) \equiv \{\{q_s\} : q_1^*(t_j) - \delta(t_j) \leqslant q_1 \leqslant q_1^*(t_j) + \delta(t_j),$$

$$q_2^*(t_j) - \delta(t_j) \leqslant q_2 \leqslant q_2^*(t_j) + \delta(t_j)\}$$

The components $g_s(t_j, t_{j-1})$ of the vector $g(t_j, t_{j-1})$ to which the domain $G_\vartheta(t_{j-1})$ shifts by the instant $t = t_j$ are of the form

$$g_s(t_j, t_{j-1}) = \int_{t_{j-1}}^{t_j} \operatorname{sh}(\vartheta - t_j)\, u_s^{(j-1)}(\tau)\, d\tau$$

The domain $G_\vartheta(t_j)$ is therefore given by the inequalities

$$G_\vartheta(t_j) \equiv \{\{q_s\}:\quad a_s^{(j)} \leqslant q_s \leqslant b_s^{(j)}\}$$

$$a_s^{(j)} = \max\,(a_s^{(j-1)} + g_s(t_j, t_{j-1}),\ q_s^*(t_j) - \delta(t_j))$$

$$b_s^{(j)} = \min\,(b_s^{(j-1)} + g_s(t_j, t_{j-1}),\ q_s^*(t_j) + \delta(t_j))$$

The control $u^*(t)$ $(t_j \leqslant t \leqslant \vartheta)$ can be found by solving inequality (2.7) and then applying maximum condition (2.8). We can therefore proceed on the basis of simple geometric relations. Specifically, the vector k° and the quantity ζ_j° at each instant $t = t_j$ can be readily determined from the condition of adjacency of the attainability domain $\Gamma[0, \mu, t_j, \vartheta]$ and the level surfaces

$$\gamma\,[\{w\}_m] = \max_q\ (|\,q_1 + \{w_1^{(j)}(\vartheta)\}_m\,|,\ |\,q_2 + \{w_2^{(j)}(\vartheta)\}_m\,|), \qquad q \in G_\vartheta(t_j)$$

The attainability domain $\Gamma[0, \mu, t_j, \vartheta]$ is in this case a disk of the radius

$$r^2 = \mu\ (\operatorname{ch}(\vartheta - t_j) - 1)$$

The level surfaces $\gamma\ [\{w\}_m] \leqslant \zeta$ in this case are rectangles (Fig. 2) with the sides

$$q_1' = b_1^{(j)} - \zeta, \quad q_1'' = a_1^{(j)} + \zeta, \quad q_2' = b_2^{(j)} - \zeta, \quad q_2'' = a_2^{(j)} + \zeta$$

$$\zeta \geqslant {}^1\!/_2 \max\,(\,|\,b_1^{(j)} - a_1^{(j)}\,|,\ |\,b_2^{(j)} - a_2^{(j)}\,|).$$

The vector k° is therefore the unit vector of the normal to the surface $\Gamma[0, \mu, t_j, \vartheta]$. Maximum condition (2.8) implies that the control $u^*(t)$ is constant and equal to $u^*(t) = -\mu k^\circ$ over the entire interval $t_j \leqslant t \leqslant \vartheta$.

Numerical realization of the above procedure for the values

$\mu = 5$, $\nu = 0{,}2$, $\vartheta = 2$, $x(0) = (2, 1, 4, 0)$, $a_1^{(0)} = 10$, $a_2^{(0)} = 14$, $b_1^{(0)} = b_2^{(0)} = 16$ and for the correction instants

$$t_0 = 0, \quad t_1 = 0.5, \quad t_2 = 1.0, \quad t_3 = 1.25, \quad t_4 = 1.5$$

yielded the following values of the controlling forces:

$$u^{(0)} = \{-3.536,\ -3.536\},\ u^{(1)} = \{-2.096,\ -4.539\},\ u^{(2)} = \{-1.462, -4.782\},\ u^{(3)} = \{-0.441,\ -4{,}980\},\ u^{(4)} = \{-2.618,\ -4.259\}$$

and the following values for the misses $J(G(t_j)) = \zeta_j^0$:

$$\zeta_0^\circ = 6.233,\ \zeta_1^\circ = 4.879,\ \zeta_2^\circ = 4.314,\ \zeta_3^\circ = 4{,}252,\ \zeta_4^\circ = 3.975$$

We note that if the value of the vector $q(0)$ were known exactly from the start (in this case $q(0) = \{11.151,\ 15.048\}$), then the control $u(t)$ (4.4) (in this case $u(t) = \{-2.758,\ -4.132\}$) would yield the miss $\varepsilon(\vartheta) = 3.531$.

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A MODIFIED CRITERION OF INSTABILITY OF MOTION

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Ideas related to Liapunov's second method and developed by Chetaev [e.g. see [1 and 3]) are used to obtain an instability criterion somewhat different from the well-known criteria of Liapunov and Chetaev.

1. Introduction. Let us consider the stability of the equilibrium position of a system of ordinary differential equations

$$dx \,/\, dt = X(x, t), \quad X(0, t) = 0 \tag{1.1}$$

We assume that the n-dimensional vector function $X(x, t)$ is continuous in t and has continuous partial derivatives with respect to x.

The criterion which we shall formulate is valid for systems of any order m; in order to illustrate the basic ideas geometrically, we shall first sonsider a simple example of a third-order system.

Let us assume that for $\xi > 0$ (here and below ξ, η, ζ represent the components of the vector x) we have the inequalities

$$\frac{d\xi}{dt} \geqslant \delta(\xi) > 0, \qquad \text{if} \quad \max(|\eta|, |\zeta|) \leqslant k\xi \tag{1.2}$$

$$\frac{d}{dt}(|\eta| - k\xi) < 0 \quad \text{for} \ |\eta| = k\xi, \ |\zeta| \leqslant k\xi \tag{1.3}$$

$$\frac{d}{dt}(|\zeta| - k\xi) > 0 \quad \text{for} \ |\zeta| = k\xi, \ |\eta| \leqslant k\xi \tag{1.4}$$

Let us consider the pyramid $OA'B'C'D'$ (see Fig. 1) defined by the inequality max $(|\eta|, |\zeta|) \leqslant k\xi$ and intersected by the plane $ABCD (\xi = \varepsilon)$ in the immediate neighborhood of the origin. Conditions (1.2)-(1.4) imply that the trajectories enter the truncated pyramid T, intersecting its surface at points belonging to the portion S_1 consisting of the faces $ABCD$, $BB'C'C$ and $DD'A'A$; similarly, the trajectories emerge from the pyramid through points of the portion S_2 of its surface consisting of the faces $BB'A'A$, $DD'C'C$ and $A'B'C'D'$. As we see from condition (1.2), the representing point can lie inside T for a finite time only; this implies that the family of trajectories entering the pyra-

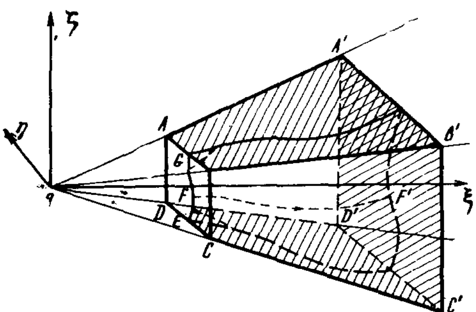

Fig. 1